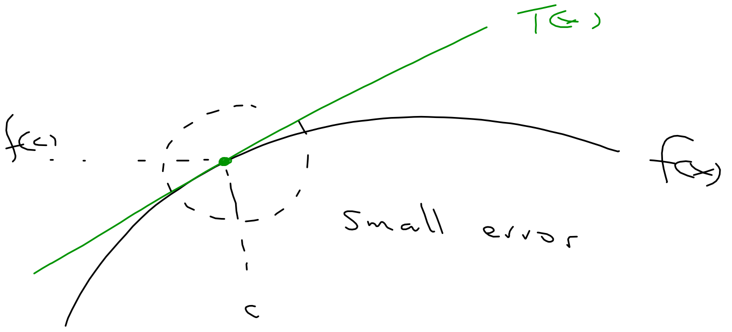
4.8 Linear approximation



A tangent line $T(x)$ to the curve of a function $f(x)$ at $x=c$ can be used to approximate the function $f(x)$; we write $f(x) \simeq T(x)$. Moreover, the value of f as $x \to c$ (as x approaches c) are very close to those of $T(x)$ as $x \to c$

Formely, If f is differentiable at $x=c$ then, the linearization of f at the values of x close to c is

$$\boxed{f(x) \simeq T(x) = f(c) + f'(c)(x-c)}$$

Ex: Linearize $f(x) = \sqrt{x}$ at $c=1$

Solution:

- $f(c) = f(1) = \sqrt{1} = 1$
- $f'(x) = \left[x^{\frac{1}{2}}\right]' = \frac{1}{2\sqrt{x}} \Longrightarrow f'(1) = \frac{1}{2}$
- Tangent line, $T(x)$ approximates $f(x) = \sqrt{x}$ at $x=1$, so we have:

$$T(x) = f(1) + f'(1)(x-1) = 1 + \frac{1}{2}(x-1) = \frac{1}{2}x + \frac{1}{2}$$

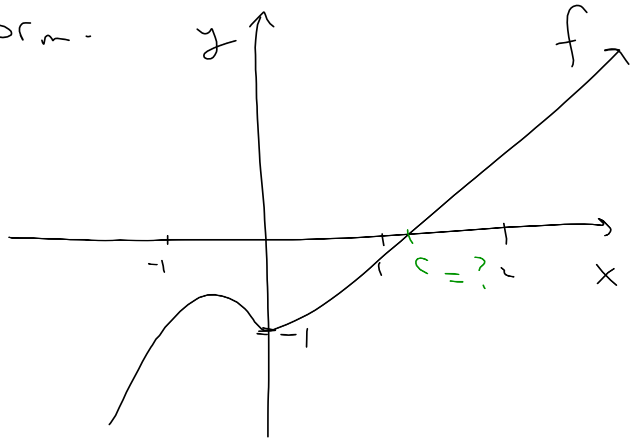
So
$$\boxed{f(x) \simeq T(x) = \frac{1}{2}x + \frac{1}{2}}$$

approximated by

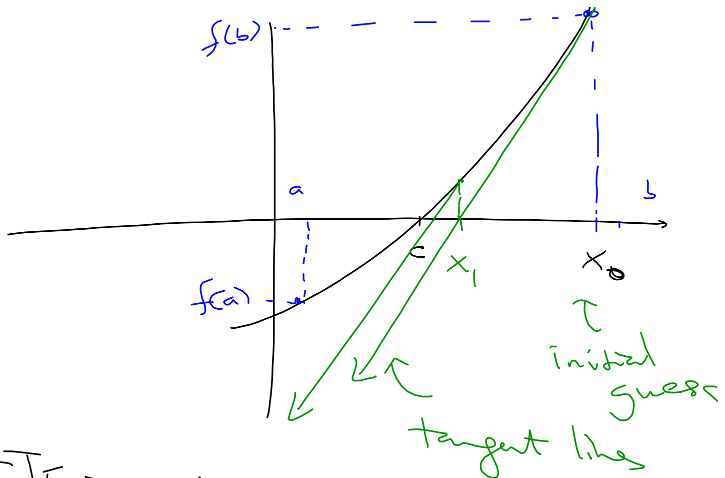
3.8 Newton's Method

Procedure to approximate the zeros (or roots or x-intercepts) of a function using tangent lines in sequence.

For instance $f(x) = x^5 - x - 1$ has only one root and its root has no algebraic form.



The newton's method allows us to approximate c



Mean Value Theorem: indicates that since $f(a)\cdot f(b)<0$ there is a zero $a<c<b$

STEPS: Newton's method:

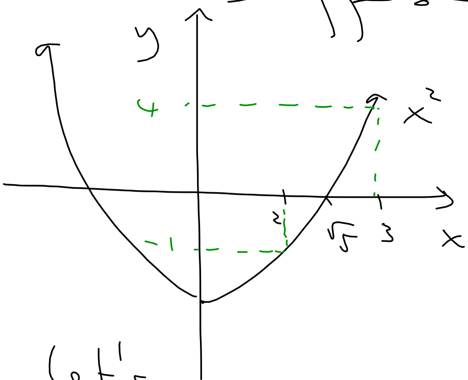
1. Initial guess x_0 (sometimes given)
2. Compute iteratively, $x_1, x_2, x_3, \ldots$

where
$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

EX: Approximating $\sqrt{5}$

Solution:

Suppose $f(x) = x^2 - 5$



Clearly the zero of $f(x)$ is $x^2 - 5 = 0$

$x = \pm\sqrt{5}$

Let's assume our initial guess is $x_0 = 2$

Let's find x_1, x_2 and x_3

$$f(x) = x^2 - 5 \longrightarrow f'(x) = 2x$$

$$x_1 = 2 - \frac{(2^2 - 5)}{2(2)} \simeq 2.25$$

$$x_2 = x_1 - \frac{x_1^2 - 5}{2x_1} = 2.25 - \frac{[(2.25)^2 - 5]}{[2(2.25)]} \simeq 2.236$$

$$x_3 = x_2 - \frac{x_2^2 - 5}{2(x_2)} = 2.236 - \frac{[(2.236)^2 - 5]}{[2(2.236)]}$$

$$\simeq 2.23607$$

Note: $\sqrt{5} \simeq 2.2236067$

Our estimate of $\sqrt{5}$ is correct up to 4 decimals